B.Sc. Part I 2nd Paper

Integral Calculus

Reduction formula

Q. Evaluate $\int_0^{\pi/2} \cos^n x \, dx$.

Soln Let $I_n = \int_0^{\pi/2} \cos^n x \, dx \quad \text{——— (1)}$

$$\Rightarrow I_n = \int_0^{\pi/2} \underbrace{\cos^{n-1} x}_{U} \; \underbrace{\cos x}_{V} \, dx$$

$$\Rightarrow I_n = \left[\cos^{n-1} x \cdot \int \cos x \, dx\right]_0^{\pi/2} - \int_0^{\pi/2}\left[\frac{d}{dx}(\cos^{n-1} x)\int \cos x \, dx\right]dx$$

$$\Rightarrow I_n = \left[\cos^{n-1} x \; \sin x\right]_0^{\pi/2} + (n-1)\int_0^{\pi/2} \cos^{n-2} x \cdot \sin x \cdot \sin x \, dx$$

$$\Rightarrow I_n = 0 + (n-1)\int_0^{\pi/2} \cos^{n-2} x \cdot \sin^2 x \, dx$$

$$\Rightarrow I_n = (n-1)\int_0^{\pi/2} \cos^{n-2} x \cdot (1-\cos^2 x) \, dx$$

$$\Rightarrow I_n = (n-1)\int_0^{\pi/2} (\cos^{n-2} x - \cos^n x) \, dx$$

$$\Rightarrow I_n = (n-1)\int_0^{\pi/2} \cos^{n-2} x\,dx - (n-1)\int_0^{\pi/2} \cos^n x\,dx$$

$$\Rightarrow I_n = (n-1) I_{n-2} - (n-1) I_n$$

$$\Rightarrow I_n + (n-1) I_n = (n-1) I_{n-2}$$

$$\Rightarrow n I_n = (n-1) I_{n-2}$$

$$\Rightarrow I_n = \frac{n-1}{n} I_{n-2} \quad \text{———— (2)}$$

Replacing n by $n-2$, we get

$$I_{n-2} = \frac{n-3}{n-2} I_{n-4} \quad \text{———— (3)}$$

Replacing n by $n-4$, in (2) we get

$$I_{n-4} = \frac{n-5}{n-4} I_{n-6} \quad \text{———— (4)}$$

In (2), replacing n by n by $n-6$, we get

$$I_{n-6} = \frac{n-7}{n-6} I_{n-8} \quad \text{———— (5)}$$

using (3), (4), (5) in (2), we get

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdot \frac{n-7}{n-6} I_{n-8}$$

$$\Rightarrow I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \frac{n-5}{n-4} \frac{n-7}{n-6} I_{n-8} \quad \text{---(6)}$$

Case I If n be odd then

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \frac{n-5}{n-4} \frac{n-7}{n-6} \cdots \frac{2}{3} I_1 \quad \text{---(7)}$$

Putting $n = 1$ in (1), we get

$$I_1 = \int_0^{\pi/2} \cos x \, dx = \left[\sin x\right]_0^{\pi/2}$$

$$= 1$$

Putting this value in (7), we get

$$\Rightarrow \boxed{I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \frac{n-5}{n-4} \cdots \frac{2}{3}, \quad n = \text{odd}}$$

Case II If n be even then, from (6)

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdot \frac{n-7}{n-6} \cdots \frac{1}{2} \cdot I_0 \quad \text{---(8)}$$

Putting $n=0$ in eq (1), we get

$$\therefore I_n = \int_0^{\pi/2} \cos^n x \, dx$$

$$\Rightarrow \quad I_0 = \int_0^{\pi/2} (\cos x)^0 \, dx = \int_0^{\pi/2} dx = \frac{\pi}{2}$$

Putting this value in (8), we have

$$\boxed{I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{1}{2} \cdot \frac{\pi}{2} ; \; n = \text{even}}$$